SMALL DEHN SURGERY AND SU(2)

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ABSTRACT. We prove that the fundamental group of 3-surgery on a nontrivial knot in S^3 always admits an irreducible SU(2)-representation. This answers a question of Kronheimer and Mrowka dating from their work on the Property P conjecture. An important ingredient in our proof is a relationship between instanton Floer homology and the symplectic Floer homology of genus-2 surface diffeomorphisms, due to Ivan Smith. We use similar arguments at the end to extend our main result to infinitely many surgery slopes in the interval [3, 5).

1. INTRODUCTION

The SU(2)-representation variety

 $R(Y) = \operatorname{Hom}(\pi_1(Y), SU(2))$

associated with a 3-manifold Y is an object of central importance in instanton gauge theory. A basic question about such varieties is whether they contain irreducible representations. A homomorphism into SU(2) is reducible if and only if it has abelian image, inspiring the following definition:

Definition 1.1. A 3-manifold Y is SU(2)-abelian if every $\rho \in R(Y)$ has abelian image.¹ In particular, R(Y) contains an irreducible if and only if Y is not SU(2)-abelian.

The classification of SU(2)-abelian 3-manifolds is a wide-open problem, even among Dehn surgeries on knots.² In their proof of the Property P conjecture [KM04b], Kronheimer and Mrowka proved that if K is a nontrivial knot in S^3 then $S_1^3(K)$ is not SU(2)-abelian (hence, not a homotopy sphere). They then strengthened this result in [KM04a], proving under the same hypothesis that $S_r^3(K)$ is not SU(2)-abelian for any rational number $r \in [0, 2]$.

It is natural to ask whether this also holds for larger values of r. Kronheimer and Mrowka explicitly posed this question in [KM04a] for the next two integers, r = 3 and 4, noting that it fails for r = 5 because 5-surgery on the right-handed trefoil is a lens space. Baldwin and Sivek answered half of this question in [BS19], proving that $S_4^3(K)$ is not SU(2)-abelian for any nontrivial knot $K \subset S^3$. Our main theorem answers the remaining half:

Theorem 1.2. $S_3^3(K)$ is not SU(2)-abelian for any nontrivial knot $K \subset S^3$.

Remark 1.3. We ultimately expect that r-surgery on a nontrivial knot is not SU(2)-abelian for any rational number $r \in [0, 5)$. Our techniques can be used to prove this for infinitely many additional slopes in the interval [3, 5), as discussed in §1.2.

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¹If $b_1(Y) = 0$ then ρ has abelian image if and only if it has cyclic image.

²Though see [LPCZ21, BS21] and [SZ19] for progress on this problem in the cases of toroidal homology spheres and non-hyperbolic geometric manifolds, respectively.

Our proof involves a novel blend of ingredients, including a relationship between instanton Floer homology and the symplectic Floer homology of genus-2 surface diffeomorphisms, due to Smith [Smi12]; ideas of Baldwin–Hu–Sivek from their recent work on cinquefoil detection [BHS21]; and new results of Li–Ye relating the graded Euler characteristics of instanton and Heegaard knot Floer homology [LY21b, LY21a]. Along the way, we partially characterize instanton L-space knots of genus 2 (Theorem 2.1 and Remark 2.6), and prove several other results (Proposition 3.2, and Theorems 4.1 and 5.1) which may be of independent interest. We outline our proof of Theorem 1.2 in detail below.

1.1. **Proof outline.** Suppose $K \subset S^3$ is a nontrivial knot. Let us assume for a contradiction that $S_3^3(K)$ is SU(2)-abelian. This surgered manifold is thus an instanton L-space [BS18b]. It then follows from [BS19] that K is fibered and strongly quasipositive of genus 1 or 2. If the genus is 1, then K is the right-handed trefoil, but 3-surgery on this trefoil is Seifert fibered with base orbifold $S^2(2,3,3)$ and is therefore not SU(2)-abelian, by [SZ19], a contradiction. We similarly rule out the possibility that K is the cinquefoil $T_{2,5}$.

Thus, $K \not\cong T_{2,5}$ is a genus-2 instanton L-space knot. Recent work of Li and Ye [LY21c] then implies that K has Alexander polynomial

$$\Delta_K(t) = t^2 - t + 1 - t^{-1} + t^{-2}$$

from which we conclude as in [BHS21] that K is a hyperbolic knot. Let (S, h) be an abstract open book corresponding to the fibration associated with K. The hyperbolicity of K implies that the monodromy h is freely isotopic to a pseudo-Anosov homeomorphism $\psi: S \to S$.

In [Smi12], Smith proved that (a version of) the instanton Floer homology of the mapping torus of a genus-2 surface diffeomorphism encodes the symplectic Floer cohomology of the diffeomorphism. We use Smith's result, together with a calculation of the framed instanton homology $I^{\#}(S_0^3(K), \mu)$, to prove that the homeomorphism ψ has no fixed points, following ideas in [BHS21]. We then apply results from [BHS21] to conclude that K is doubly periodic with unknotted quotient A and axis B, where A and B have linking number 5.

We may therefore express $S_3^3(K)$ as the branched double cover of the lens space

$$L = S^3_{3/2}(A) \cong L(3,2),$$

branched along the primitive knot $J \subset L$ induced by B in this surgery on A,

(1.1)
$$S_3^3(K) \cong \Sigma(L, J).$$

Under the assumption that $S_3^3(K)$ is SU(2)-abelian, we prove that every representation

$$\rho: \pi_1(L \setminus N(J)) \to SU(2)$$

which sends a fixed meridian of J to i has finite cyclic image. There are exactly $|H_1(L)| = 3$ such representations. We prove that these representations are nondegenerate as generators of the instanton knot homology of J, and thereby conclude that J is an *instanton Floer* simple knot, meaning that

$$\dim_{\mathbb{C}} KHI(L, J) = \dim_{\mathbb{C}} I^{\#}(L) = 3.$$

,,

We deduce that J is rationally fibered, and that the graded Euler characteristic of KHI(L, J) is given by

$$\chi_{\rm gr}(KHI(L,J)) = t^{-n} + 1 + t^n,$$

for some n = g(F) + 1, where F is a minimal-genus rational Seifert surface for J.

In recent work [LY21b], Li and Ye proved an equality between the graded Euler characteristics of instanton knot homology and (Heegaard) knot Floer homology,

$$\chi_{\rm gr}(\widehat{H}F\widehat{K}(L,J)) = \pm \chi_{\rm gr}(KHI(L,J)).$$

We combine this with constraints on the graded Euler characteristic of knot Floer homology to conclude that n = 1. Thus, J has (rational) genus 0, which implies that its fibered exterior is a solid torus. It follows that $\Sigma(L, J)$, and hence 3-surgery on K, is a lens space. Since K is doubly periodic and has a lens space surgery, a result of Wang and Zhou [WZ92] says that K must be a torus knot, a contradiction.

1.2. Other small surgeries. As mentioned above, we suspect that if $K \,\subset S^3$ is a nontrivial knot, then $S_r^3(K)$ is not SU(2)-abelian for any rational $r \in [0, 5)$. Long after Kronheimer and Mrowka's proof for $r \in [0, 2]$, Baldwin and Sivek proved this [BS19] for the dense subset of the interval (2, 3) consisting of rationals with prime power numerators. Proving it for the corresponding subset of [3, 5) is much harder, as explained in §6. Indeed, our main theorem concerns the single case r = 3. Nevertheless, our techniques can be used to extend Theorem 1.2 to infinitely many surgery slopes in both [3, 4) and [4, 5), and even to a special dense subset of slopes in (5, 7). Our main result in this vein is:

Theorem 1.4. Let p and q be coprime positive integers with $p/q \in [3,7)$, where p is a prime power. If either

- p is even,
- p is odd and gcd(p,5) = 1 and $p/q \in [4,5)$, or
- p is odd and gcd(p,5) = 1 and $p/q \in [3,4)$ with $23p 9 \le 80q \le 25p + 9$,

then $S^3_{n/q}(K)$ is not SU(2)-abelian for every nontrivial knot $K \subset S^3$.

The inequality in Theorem 1.4 is satisfied for $p/q \in [16/5, 80/23) \subset [3, 4)$. It is satisfied for p/q = 3 as well, so this theorem recovers Theorem 1.2. Note that Theorem 1.4 also applies to the slopes 4, 7/2, and 9/2. Together with the results mentioned above, concerning the intervals [0, 2] and (2, 3), this proves that:

Corollary 1.5. If $K \subset S^3$ is a nontrivial knot then $S^3_r(K)$ is not SU(2)-abelian for any integer or half-integer r < 5.

1.3. **Organization.** In §2, we prove that if $K \not\cong T_{2,5}$ is an instanton L-space knot of genus 2, then K is doubly-periodic with unknotted quotient. We then use this in §3 to show that if 3-surgery on a nontrivial knot is SU(2)-abelian, then this surgery is the branched double cover of a knot $J \subset L(3,2)$; in fact, we prove something more general (Proposition 3.2). In §4, we prove that such J are instanton Floer simple (Theorem 4.1). In §5, we combine these results with an understanding of instanton Floer simple knots in lens spaces (Theorem 5.1) to prove Theorem 1.2. Finally, in §6, we extend our main theorem to other surgery slopes in [3, 5), proving Theorem 1.4.

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2. Genus two L-space knots

Recall that a rational homology 3-sphere Y is an *instanton L-space* if its framed instanton homology satisfies

$$\lim_{\mathbb{C}} I^{\#}(Y) = |H_1(Y)|.$$

A knot $K \subset S^3$ is said to be an *instanton L-space knot* if $S_r^3(K)$ is an instanton L-space for some rational number r > 0. Such knots were studied extensively in [BS19]; in particular, it was shown there [BS19, Theorem 1.15] that instanton L-space knots are fibered and strongly quasipositive.

Our main goal in this section is to prove the following:

Theorem 2.1. Let $K \subset S^3$ be an instanton L-space knot of genus 2. Then K is fibered and strongly quasipositive, with Alexander polynomial

$$\Delta_K(t) = t^2 - t + 1 - t^{-1} + t^{-2}.$$

If $K \not\cong T_{2,5}$, then there exists a pseudo-Anosov 5-braid β whose closure $B = \hat{\beta}$ is an unknot with braid axis A, such that K is the lift of A in the branched double cover

$$\Sigma(S^3, B) \cong S^3.$$

In particular, K is a doubly periodic knot with unknotted quotient A and axis B.

A Heegaard Floer analogue of this result was proved in [BHS21, §3]. Our proof of Theorem 2.1 will largely follow the arguments there, and we will refer to specific parts of [BHS21] for steps whose proofs are the same in either setting. We begin with some preliminary results.

Lemma 2.2. Let K be an instanton L-space knot of genus 2. Then K is fibered and strongly quasipositive, with Alexander polynomial

$$\Delta_K(t) = t^2 - t + 1 - t^{-1} + t^{-2},$$

and either $K \cong T_{2,5}$ or K is hyperbolic.

Proof. The claims that K is fibered and strongly quasipositive are [BS19, Theorem 1.15], as discussed above. The Alexander polynomial $\Delta_K(t)$ is determined by [LY21c, Theorem 1.9], which says that there are integers $k \geq 0$ and

$$n_k > n_{k-1} > \dots > n_0 = 0 > n_{-1} > \dots > n_{-k},$$

with $n_{-i} = -n_i$ for all j, such that the instanton knot homology KHI(K) satisfies

$$\dim_{\mathbb{C}} KHI(K, i) = \begin{cases} 1, & i = n_j \text{ for some } j \\ 0, & \text{otherwise,} \end{cases}$$

with the nonzero summands $KHI(K, n_j)$ supported in alternating $\mathbb{Z}/2\mathbb{Z}$ gradings. We know that $KHI(K, 2) \neq 0$ since K has genus 2 [KM10b, Proposition 7.16], and then $KHI(K, 1) \neq 0$ o since K is fibered [BS18a, Theorem 1.7], so k = 2 and $(n_2, n_1, n_0) = (2, 1, 0)$. Then KHI(K) has graded Euler characteristic

$$\sum_{j \in \mathbb{Z}} \chi(KHI(K,j)) \cdot t^j = \pm \Delta_K(t)$$

[KM10a, Lim10], so the normalization $\Delta_K(1) = 1$ forces $\Delta_K(t)$ to be as claimed. Since K is fibered with this particular Alexander polynomial, it is either $T_{\pm 2,5}$ or hyperbolic by [BHS21, Lemma 3.2]. The case of $T_{-2,5}$ is ruled out as this knot is not strongly quasipositive. \Box

Lemma 2.3. Let K be an instanton L-space knot of genus 2, and let $w \to S_0^3(K)$ be a Hermitian line bundle such that $c_1(w)$ is Poincaré dual to the meridian μ of K. Then

$$\dim_{\mathbb{C}} I_*(S^3_0(K))_w = 6$$

where I_* refers here to the version of instanton homology for admissible bundles described, for example, in [Don02, §5.6].

Proof. We know that 3-surgery on K is an instanton L-space [BS19, Proposition 7.11], while neither 1- nor 2-surgery on K is an instanton L-space [BS19, Proposition 7.12]. It follows in the language of [BS20] that $\nu^{\sharp}(K) = r_0(K) = 3$, meaning that

$$\dim_{\mathbb{C}} I^{\#}(S^{3}_{p/q}(K)) = \begin{cases} p, & p/q \ge 3\\ 6q - p, & p/q < 3 \end{cases}$$

for all coprime integers p, q with $q \ge 1$. In particular we have

$$\dim_{\mathbb{C}} I^{\#}(S_0^3(K),\mu) = 6$$

by [BS20, Proposition 3.3].

To prove the lemma, we then relate $I_*(S_0^3(K))_w$ to $I^{\#}(S_0^3(K), \mu)$ using Fukaya's connected sum theorem for instanton homology [Sca15, Theorem 1.3]. To be precise, since $H_2(S_0^3(K))$ is generated by a closed surface of genus 2 which is dual to μ , the special case of this theorem discussed in [Sca15, §9.8] says that

$$I^{\#}(S_0^3(K),\mu) \otimes H_*(S^4) \cong I_*(S_0^3(K))_w \otimes H_*(S^3)$$

as relatively $\mathbb{Z}/4\mathbb{Z}$ -graded vector spaces over any field of characteristic zero. It follows that $\dim_{\mathbb{C}} I_*(S^3_0(K))_w = 6$ as claimed.

Recall for the proposition below that instanton L-space knots are fibered.

Proposition 2.4. Let $K \not\cong T_{2,5}$ be an instanton L-space knot of genus 2. Let (S,h) be an open book encoding the fibration of its complement. Then the monodromy h is freely isotopic to a pseudo-Anosov homeomorphism

$$\psi: S \to S$$

with no fixed points.

Proof. The complement $S^3 \setminus K$ is hyperbolic, by Lemma 2.2. It follows that the monodromy h is freely isotopic to a pseudo-Anosov homeomorphism

$$\psi: S \to S$$

by work of Thurston [Thu98]. Since K is strongly quasipositive, also by Lemma 2.2, [BHS21, Lemma 3.3] says that the stable foliation of ψ has some number $n \ge 2$ of boundary prongs.

As discussed in [BHS21, §2], the fact that $n \ge 2$ means that the map ψ extends naturally to a pseudo-Anosov homeomorphism

$$\hat{\psi}: \hat{S} \to \hat{S}$$

of the closed genus-2 surface obtained from S by capping off its boundary with a disk. The stable foliation of ψ extends across the disk to a stable invariant foliation for $\hat{\psi}$ in which the n boundary prongs extend to n prongs meeting at a singularity p in the capping disk (except that p is a smooth point if n = 2). Moreover, p is a fixed point of $\hat{\psi}$.

Note that the mapping torus $Y_{\hat{\psi}}$ of $\hat{\psi}$ is homeomorphic to 0-surgery on K,

$$Y_{\hat{\psi}} \cong S^3_0(K),$$

by a map which identifies the suspension of p with the meridian μ of K in the 0-surgery. Smith [Smi12, Corollary 1.8] proved in this case (or, more generally, for any genus-2 mapping torus which is a homology $S^1 \times S^2$ with a distinguished section coming from a marked point on the surface) that there is an isomorphism (with \mathbb{C} coefficients) of the form

(2.1)
$$HF_{inst}(E \to Y_{\hat{\psi}}) \cong \mathbb{C} \oplus HF^*_{symp}(\hat{\psi}) \oplus \mathbb{C}.$$

Here, $E \to Y_{\hat{\psi}}$ is the nontrivial SO(3) bundle with w_2 dual to μ ; $HF^*_{symp}(\hat{\psi})$ is the symplectic Floer cohomology of $\hat{\psi}$, defined via any area-preserving diffeomorphism (for any area form on \hat{S}) in the mapping class of $\hat{\psi}$; and HF_{inst} is the version of instanton homology used by Dostoglou and Salamon in [DS94]. The latter is defined using a slightly larger gauge group than the determinant-1 gauge transformations defining I_* , and is thus the quotient of the relatively $\mathbb{Z}/8\mathbb{Z}$ -graded I_* by a degree-4 involution. It follows, in particular, that

$$\dim_{\mathbb{C}} HF_{\text{inst}}(E \to Y_{\hat{\psi}}) = \frac{1}{2} \cdot \dim_{\mathbb{C}} I_*(S_0^3(K))_w;$$

where $w \to S_0^3(K)$ is a Hermitian line bundle with c_1 dual to μ , as in Lemma 2.3. The same lemma says that $\dim_{\mathbb{C}} I_*(S_0^3(K))_w = 6$, from which it follows that

$$\dim_{\mathbb{C}} HF_{\text{inst}}(E \to Y_{\hat{\psi}}) = 3,$$

and therefore that

(2.2)
$$\dim_{\mathbb{C}} HF^*_{\mathrm{symp}}(\hat{\psi}) = 1$$

by the isomorphism (2.1).

The symplectic Floer cohomology of pseudo-Anosov homeomorphisms was computed by Cotton-Clay in [CC09, §3]. In particular, given $\hat{\psi}$ as above, Cotton-Clay defines in [CC09, §3.2] a canonical smooth representative $\hat{\psi}_{sm}$ of $\hat{\psi}$ whose symplectic Floer chain complex

$$CF_*^{\text{symp}}(\hat{\psi}_{sm})$$

is freely generated by the fixed points of $\hat{\psi}_{sm}$ and has trivial differential. This implies that

(2.3)
$$HF^*_{\text{symp}}(\hat{\psi}) \cong CF^{\text{symp}}_*(\hat{\psi}_{sm}).$$

(This is proved in [CC09] with $\mathbb{Z}/2\mathbb{Z}$ coefficients, but also holds in characteristic zero since the differential vanishes for purely topological reasons.) Furthermore, $\hat{\psi}_{sm}$ has the property that each fixed point of $\hat{\psi}$ is also a fixed point of $\hat{\psi}_{sm}$. The combination of (2.2) and (2.3) therefore implies that $\hat{\psi}$ has at most one fixed point. On the other hand, we know that p is a fixed point of $\hat{\psi}$ from the discussion above. It follows that this is the only fixed point of $\hat{\psi}$, which implies that the original map ψ has no fixed points.

Remark 2.5. The proof of Proposition 2.4 is notably easier than that of the analogous [BHS21, Theorem 3.5], at least modulo deep theorems by others. The reason is that the work of Lee and Taubes [LT12] used in [BHS21] requires a monotonicity condition which only applies in genus 3 or higher, so a trick is required to pass from the genus 2 case to the higher genus case. By contrast, the references [DS94, Smi12] work perfectly well for genus-2 mapping tori, and in fact [Smi12, Corollary 1.8] only applies in genus 2.

Remark 2.6. Suppose $K \not\cong T_{2,5}$ is a genus-2 instanton L-space knot, (S, h) is an open book associated to the fibration of its complement, and $\psi : S \to S$ a pseudo-Anosov map isotopic to h. Proposition 2.4 says that ψ has no fixed points. Thus we can apply [BHS21, Proposition 3.8], whose other hypotheses follow from Lemma 2.2, to conclude the following:

- the monodromy h has fractional Dehn twist coefficient $c(h) = \frac{1}{4}$;
- the stable foliation of ψ has four prongs on ∂S , and two interior 3-pronged singularities which are exchanged by ψ .

We will not need either of these properties in this article, but this may be independently useful towards a characterization of genus-2 instanton L-space knots.

At this point, the proof of Theorem 2.1 proceeds in exactly the same way as that of its Heegaard Floer analogue [BHS21, Theorem 3.1].

Proof of Theorem 2.1. The first few claims follow immediately from Lemma 2.2. To argue that if $K \not\cong T_{2,5}$ then there is a pseudo-Anosov 5-braid β with unknotted closure $B = \hat{\beta}$ such that K is the lift of the braid axis A to the branched double cover of B, we simply repeat the proof of [BHS21, Theorem 3.1] verbatim. That proof relies only on the conclusions of Proposition 2.4, so it applies equally well here.

3. SU(2)-Abelian surgeries and branched double covers

The conditions of being an instanton L-space and being SU(2)-abelian are closely related. For surgeries on knots, we have the following:

Lemma 3.1. Let $K \subset S^3$ be a knot, and suppose p and q are positive coprime integers such that p is a prime power. If

$$Y = S_{p/q}^3(K)$$

is SU(2)-abelian, then Y is an instanton L-space.

Proof. An SU(2)-abelian rational homology sphere Y is an instanton L-space whenever $\pi_1(Y)$ is cyclically finite, by [BS18b, Theorem 4.6]. This cyclical finiteness is satisfied if, for example, $H_1(Y)$ is cyclic of prime power order, by [BS18b, Proposition 4.9].

Our main goal is to prove that 3-surgery on a nontrivial knot in S^3 is never SU(2)-abelian. In this section we use Lemma 3.1, together what we proved about genus-2 instanton L-space knots in §2, to reduce the problem to one about branched double covers of knots in the lens space L(3, 2), as outlined in the introduction. In fact, we prove a substantially more general result in Proposition 3.2 below. We will use this additional generality in §6 to address other surgeries.

Proposition 3.2. Let $K \subset S^3$ be a nontrivial knot, and suppose p, q are coprime positive integers with p/q < 5, where p is an odd prime power. If

$$Y = S^3_{p/q}(K)$$

is SU(2)-abelian, then:

- Y is not Seifert fibered,
- K is hyperbolic, and doubly periodic with unknotted quotient A, and

• Y is homeomorphic to a branched double cover of the lens space

$$L = S^3_{p/2q}(A) = L(p, 2q)$$

along some knot $J \subset L$ which is homologous to 5 times the core of the solid torus $S^3 \setminus N(A) \subset L$ in the genus-1 Heegaard splitting of L along the surface $\partial N(A)$.

Proof. Lemma 3.1 implies that Y is an instanton L-space; in particular, K is an instanton L-space knot. It then follows from [BS19, Theorem 1.15] that

$$2g(K) - 1 \le p/q < 5,$$

which implies that K has genus 1 or 2.

We next observe that K is neither $T_{2,3}$ nor $T_{2,5}$. Indeed, p/q < 5 implies that

|abq - p| > 1, for (a, b) = (2, 3) and (2, 5).

Then (p/q)-surgeries on $T_{2,3}$ and $T_{2,5}$ are Seifert manifolds which are not lens spaces [Mos71]. Such surgeries cannot be SU(2)-abelian [SZ19, Remark 1.3]. We conclude that K is neither $T_{2,3}$ nor $T_{2,5}$. Since $T_{2,3}$ is the only genus-1 instanton L-space knot [BS19, Proposition 7.12], it follows that K must be a genus-2 instanton L-space knot other than $T_{2,5}$.

Theorem 2.1 then says that K is hyperbolic, and doubly periodic with unknotted quotient A and axis B, where A and B have linking number 5. In particular, there is an involution $\tau: S^3 \to S^3$ with $\tau(K) = K$ whose fixed set is the axis B. This involution extends from the exterior $S^3 \setminus N(K)$ across the solid torus realizing the Dehn surgery producing

$$Y = S^3_{p/q}(K),$$

and it acts freely on this solid torus since p is odd. Thus, τ induces an involution on Y with fixed set B; the quotient of Y by this action is then the lens space

$$L = S^3_{p/2q}(A) \cong L(p, 2q)$$

obtained as (p/2q)-surgery on the unknotted quotient A. In particular, Y is homeomorphic to a branched double cover of L along the image J of B in this lens space. The fact that Aand B have linking number 5 implies that J is homologous in $H_1(L)$ to 5 times the core of the solid torus $S^3 \setminus N(A)$ in the genus-1 Heegaard splitting of L along the surface $\partial N(A)$.

Finally, we argue that

$$Y = S_{n/a}^3(K)$$

is not Seifert fibered. Suppose that it is. Then Y is a lens space, by [SZ19, Remark 1.3]. In this case, since K is doubly periodic (and thus admits a symmetry with 1-dimensional fixed point set which is not a strong inversion) and has a nontrivial cyclic surgery, a theorem of Wang and Zhou [WZ92, Proposition 3] says that K must be a torus knot, a contradiction.

4. BRANCHED DOUBLE COVERS AND FLOER SIMPLE KNOTS

Our goal in this section is to prove Theorem 4.1 below. This result, in combination with Proposition 3.2, implies that if 3-surgery on a nontrivial knot is SU(2)-abelian, then the surgered manifold is the branched double cover of an instanton Floer simple knot in L(3, 2). We will combine this with an understanding of Floer simple knots in this lens space in §5 to complete the proof of Theorem 1.2, as outlined in the introduction, and we will apply Theorem 4.1 to infinitely many additional surgery slopes in §6. The various results in this section may be of independent interest.

Theorem 4.1. Suppose L is a rational homology sphere in which $H_1(L)$ is cyclic of order an odd prime power, and let $J \subset L$ be a primitive knot. If the branched double cover $\Sigma(L, J)$ is SU(2)-abelian and satisfies

$$|H_1(\Sigma(L,J))| = |H_1(L)|,$$

then

$$\dim_{\mathbb{C}} KHI(L,J) = |H_1(L)|.$$

In other words, the knot $J \subset L$ is instanton Floer simple.

We start by clarifying some of the terminology and notation in the statement of Theorem 4.1, explaining in particular why a primitive knot has a unique branched double cover.

For the rest of this section, L will refer to a rational homology sphere, and J to a knot in L. We say that $J \subset L$ is *primitive* if its homology class [J] generates $H_1(L;\mathbb{Z})$. In this case, an exercise in algebraic topology shows that

$$H_1(L \setminus N(J)) \cong \mathbb{Z}.$$

This group is generated by the homology class of some peripheral curve $\alpha \subset \partial N(J)$. Dehn filling along α produces an integral homology sphere Z, and if C is the core of this filling, then α is a meridian μ_C of C, and we have a homeomorphism

$$Z \setminus N(C) \cong L \setminus N(J).$$

Since Dehn filling $Z \setminus N(C)$ along the meridian μ_J of J recovers L, it follows that

$$[\mu_J] = |H_1(L)| \cdot [\mu_C]$$

in the first homology of this knot complement. We will use the notation established in this paragraph throughout the rest of this section.

Lemma 4.2. Suppose $H_1(L)$ has odd order and that $J \subset L$ is primitive, and let $C \subset Z$ be as above. Then $J \subset L$ and $C \subset Z$ have unique branched double covers, $\Sigma(L,J)$ and $\Sigma(Z,C)$, which satisfy

$$H_1(\Sigma(L,J)) \cong H_1(\Sigma(Z,C)) \oplus H_1(L).$$

Proof. Since the knot $C \subset Z$ is nullhomologous, we can take a meridian μ_C and longitude λ_C in $\partial N(C)$ and write

$$\mu_J = (\mu_C)^p (\lambda_C)^q$$

in $\pi_1(\partial N(C))$ for some relatively prime integers p and q, where we reverse the orientation of C if necessary to arrange $p \ge 0$. Then $H_1(L) \cong \mathbb{Z}/p\mathbb{Z}$.

To define $\Sigma(L, J)$, we first consider the unique connected double cover $M \to L \setminus N(J)$ of $L \setminus N(J)$, specified by the kernel of the map

$$\pi_1(L \setminus N(J)) \to H_1(L \setminus N(J)) \cong \mathbb{Z} \xrightarrow{\text{mod } 2} \mathbb{Z}/2\mathbb{Z}.$$

Since $[\mu_C]$ generates $H_1(L \setminus N(J))$ and

$$[\mu_J] = |H_1(L)| \cdot [\mu_C],$$

with $|H_1(L)|$ odd, it follows that μ_J is sent to 1 under this map. Thus, μ_J^2 lifts to a simple closed curve in M. The unique branched double cover $\Sigma(L, J)$ is then formed by Dehn

filling M along this lift of μ_J^2 . We similarly define $\Sigma(Z, C)$ by Dehn filling M along a lift of μ_C^2 .

Since C is nullhomologous in Z, its lift $C \subset \Sigma(Z, C)$ is also nullhomologous, so we define $\mu_{\tilde{C}}$ and $\lambda_{\tilde{C}}$ to be its meridian and longitude in the boundary of

$$M = \Sigma(Z, C) \setminus N(\tilde{C})$$

These curves lift μ_C^2 and λ_C . If $\mu_J = (\mu_C)^p (\lambda_C)^q$ as above, then $\mu_J^2 = (\mu_C^2)^p (\lambda_C)^{2q}$ lifts to

$$\mu_{\tilde{J}} = (\mu_{\tilde{C}})^p (\lambda_{\tilde{C}})^{2q}$$

and so $\Sigma(L, J)$ is the result of (p/2q)-surgery on $\tilde{C} \subset \Sigma(Z, C)$. Since \tilde{C} is nullhomologous and $H_1(L) \cong \mathbb{Z}/p\mathbb{Z}$, this completes the proof. \Box

Remark 4.3. We will often use the shorthand $\Sigma(J)$ for $\Sigma(L, J)$ out of convenience.

The lemma below will be useful in several places.

Lemma 4.4. Suppose $H_1(L)$ has odd order and $J \subset L$ is primitive. Then

$$|H_1(\Sigma(L,J))| = |H_1(L)| \cdot |\Delta_J(-1)|,$$

where $\Delta_J(t)$ is the symmetrized Alexander polynomial of J.

Proof. Let $C \subset Z$ be as above. By [BZ03, Theorem 8.21], we have that

$$|H_1(\Sigma(Z,C))| = |\Delta_C(-1)| = |\Delta_J(-1)|,$$

where the second equality follows from the fact that C and J have homeomorphic complements and the Alexander polynomial depends only on the complement. Then

$$|H_1(\Sigma(L,J))| = |H_1(L)| \cdot |H_1(\Sigma(Z,C))| = |H_1(L)| \cdot |\Delta_J(-1)|,$$

by Lemma 4.2.

In the results below, we view SU(2) as the group of unit quaternions.

Proposition 4.5. Suppose $H_1(L)$ has odd order, that $J \subset L$ is primitive, and that

$$|H_1(\Sigma(L,J))| = |H_1(L)|.$$

Then every representation

$$\rho: \pi_1(L \setminus N(J)) \to SU(2)$$

with image in the binary dihedral group $\{e^{i\theta}\} \cup \{e^{i\theta}j\}$ actually has cyclic image. Moreover, there are exactly $|H_1(L)|$ such representations satisfying $\rho(\mu_J) = i$.

Proof. Let $C \subset Z$ be as above, so that $[\mu_C]$ generates $H_1(L \setminus N(J)) \cong \mathbb{Z}$ and

$$[\mu_J] = |H_1(L)| \cdot [\mu_C].$$

By Lemma 4.4, we have $|\Delta_J(-1)| = |\Delta_C(-1)| = 1$. Therefore $|H_1(\Sigma(C))| = 1$. It follows that there are no non-abelian representations

$$\pi_1(Z \setminus N(C)) \to SU(2)$$

with binary dihedral image, since the number of such conjugacy classes is

$$\frac{|H_1(\Sigma(C))|-1}{2}.$$

This was originally proved for knots in S^3 by Klassen [Kla91, Theorem 10], whose proof makes essential use of the Wirtinger presentation; Boden and Friedl [BF08, Corollary 1.3] give a generalization whose proof applies equally well to knots in any integral homology sphere. In any case, since $L \setminus N(J) \cong Z \setminus N(C)$, we conclude that there are no non-abelian binary dihedral representations of $\pi_1(L \setminus N(J))$ either. In particular, every binary dihedral representation of $\pi_1(L \setminus N(J))$ has image in the cyclic group $\{e^{i\theta}\}$.

If ρ is such a representation, then (since it has abelian image) it factors through a map

$$H_1(L \setminus N(J)) \cong \mathbb{Z} \to SU(2),$$

whose domain is generated by $[\mu_C]$. It follows that ρ is uniquely determined by its evaluation at μ_C , and that

$$\rho(\mu_J) = \rho(\mu_C)^{|H_1(L)|}.$$

If we require that $\rho(\mu_J) = i = e^{i\pi/2}$, then the possible ρ are specified by

$$\rho(\mu_C) = \exp\left(i \cdot \frac{\frac{\pi}{2} + 2\pi m}{|H_1(L)|}\right), \quad m = 0, 1, \dots, |H_1(L)| - 1,$$

and the proposition follows.

The following is a mild adaptation of [Zen17, Proposition 3.1].

Lemma 4.6. Suppose $H_1(L)$ has odd order, that $J \subset L$ is primitive, and that $\Sigma(L, J)$ is an SU(2)-abelian rational homology sphere. Let

$$\rho: \pi_1(L \setminus N(J)) \to SU(2)$$

be a representation satisfying $\rho(\mu_J) = i$. Then the image of

ad
$$\rho : \pi_1(L \setminus N(J)) \to SO(3)$$

is either cyclic or dihedral, and has order 2n for some odd integer n dividing $|H_1(\Sigma(J))|$.

Proof. We have a short exact sequence of groups

$$1 \to \pi_1(M) \xrightarrow{p_*} \pi_1(L \setminus N(J)) \xrightarrow{m} \mathbb{Z}/2\mathbb{Z} \to 1$$

where

$$M \xrightarrow{p} L \setminus N(J)$$

is the unique connected double covering. The class μ_J^2 belongs to ker $(m) = p_* \pi_1(M)$, since the homology class $[\mu_J^2] = 2[\mu_J]$ is zero mod 2, so we can view the normal closure $\langle\!\langle \mu_J^2 \rangle\!\rangle$ as a subgroup of $\pi_1(M)$. Upon passing to quotients, we obtain a short exact sequence

$$1 \to \pi_1(\Sigma(J)) \to \frac{\pi_1(L \setminus N(J))}{\langle\!\langle \mu_J^2 \rangle\!\rangle} \to \mathbb{Z}/2\mathbb{Z} \to 1.$$

Since $\Sigma(J)$ is SU(2)-abelian, and $H_1(\Sigma(J))$ has odd order by Lemma 4.2 and the oddness of $|H_1(L)|$, we may conclude [Zen17, Lemma 3.2] that every representation

$$\pi_1(\Sigma(J)) \to SO(3)$$

has cyclic image. (To see this, note that every representation $\pi_1(\Sigma(J)) \to SO(3)$ lifts to an SU(2) representation, since the obstruction to lifting belongs to

$$H^2(\Sigma(J); \mathbb{Z}/2\mathbb{Z}) = 0.$$

The image of this lift is abelian by assumption—and, hence, cyclic, since $b_1(\Sigma(J)) = 0$ —so the projection of this image back down to SO(3) must then be cyclic as well.) Any such representation therefore factors through $H_1(\Sigma(J))$, which is finite of odd order, and so its image also has odd order.

Now given

 $\rho: \pi_1(L \setminus N(J)) \to SU(2)$

with $\rho(\mu_J) = i$, we observe that $\operatorname{ad} \rho$ sends μ_J^2 to $\operatorname{ad}(i^2) = \operatorname{ad}(-1) = 1$, hence descends to a representation

$$\tilde{\rho}: \pi_1(L \setminus N(J)) / \langle\!\langle \mu_J^2 \rangle\!\rangle \to SO(3)$$

with $\operatorname{Im}(\tilde{\rho}) = \operatorname{Im}(\operatorname{ad} \rho)$. The short exact sequence above yields a commutative diagram

$$1 \longrightarrow \pi_1(\Sigma(J)) \longrightarrow \frac{\pi_1(L \setminus N(J))}{\langle \langle \mu_J^2 \rangle \rangle} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1$$

$$\downarrow \tilde{\rho}$$

$$SO(3)$$

in which the top row is exact, and the image of r is an odd-order cyclic subgroup of SO(3). We will write $\text{Im}(r) = \langle x \rangle$, where $x \in SO(3)$ is an element of some odd order $n \ge 1$ and n divides $|H_1(\Sigma(J))|$.

Every element of $\pi_1(L \setminus N(J)) / \langle\!\langle \mu_J^2 \rangle\!\rangle$ has the form

$$g \text{ or } \alpha \cdot g, \qquad g \in \pi_1(\Sigma(J))$$

where α is some element of the nontrivial coset of the index-2 subgroup $\pi_1(\Sigma(J))$. Thus, if we write $y = \tilde{\rho}(\alpha)$, then

$$\operatorname{Im}(\tilde{\rho}) = \langle x \rangle \cup (y \cdot \langle x \rangle).$$

If $y \in \langle x \rangle$ then $\operatorname{Im}(\tilde{\rho})$ is cyclic of order n, but this is impossible since $\operatorname{Im}(\tilde{\rho})$ contains the element $\tilde{\rho}(\mu_J) = \operatorname{ad}(i)$ of order 2. Thus, $\langle x \rangle \cong \mathbb{Z}/n\mathbb{Z}$ is an index-2 subgroup of $\operatorname{Im}(\tilde{\rho}) \subset SO(3)$, so $\operatorname{Im}(\tilde{\rho})$ has order 2n, which is not a multiple of 4. Now, every finite subgroup of SO(3) is either cyclic, dihedral, tetrahedral, octahedral, or icosahedral, and the latter three have order 12, 24, or 60, so $\operatorname{Im}(\tilde{\rho})$ must be cyclic or dihedral, of order 2n.

Proposition 4.7. Suppose $H_1(L)$ has odd order, that $J \subset L$ is primitive, and that $\Sigma(L, J)$ is an SU(2)-abelian rational homology sphere. Let

$$\rho: \pi_1(L \setminus N(J)) \to SU(2)$$

be a representation satisfying $\rho(\mu_J) = i$. Then Im(ρ) is either cyclic or binary dihedral, and has order 4n, where n divides $|H_1(\Sigma(L,J))|$. Moreover, if

$$|H_1(\Sigma(L,J))| = |H_1(L)|,$$

then ρ has cyclic image.

Proof. By Lemma 4.6, we know that the image of the composition

$$\pi_1(L \setminus N(J)) \xrightarrow{\rho} SU(2) \xrightarrow{\text{ad}} SO(3)$$

is either cyclic or dihedral, and has order 2n, where n divides $|H_1(\Sigma(J))|$ and is thus odd. Note that $\text{Im}(\rho) \subset SU(2)$ contains the kernel $\{\pm 1\}$ of the map

ad :
$$SU(2) \to SO(3)$$
,

since $\rho(\mu_J^2) = -1$. Therefore,

$$\operatorname{Im}(\rho) = \operatorname{ad}^{-1}(\operatorname{Im}(\operatorname{ad} \rho))$$

has order 4n. If $\operatorname{Im}(\operatorname{ad} \rho) \subset SO(3)$ is cyclic then so is $\operatorname{Im}(\rho) \subset SU(2)$, whereas if $\operatorname{Im}(\operatorname{ad} \rho) \subset SO(3)$ is dihedral then $\operatorname{Im}(\rho) \subset SU(2)$ is binary dihedral.

For the last assertion, if $|H_1(\Sigma(L, J))| = |H_1(L)|$ then Proposition 4.5 says that $\text{Im}(\rho)$ cannot be non-abelian and binary dihedral, so it must be cyclic.

The instanton knot homology KHI(L, J) is half of an instanton homology group defined as the Morse homology of a Chern–Simons functional on an associated space of connections. As discussed in [KM10b, §7], the space of critical points of this functional can be identified with a double cover of the representation variety

$$R(J) = \{\rho : \pi_1(L \setminus N(J)) \to SU(2) \mid \rho(\mu_J) = i\}$$

If every element of R(J) has cyclic image, then this is a trivial double cover. If in addition each $\rho \in R(J)$ is nondegenerate (corresponds to nondegenerate critical points of the Chern– Simons functional in a suitable sense), then it follows that

(4.1)
$$\dim_{\mathbb{C}} KHI(L,J) \le |R(J)|.$$

We would thus like to know when each of the cyclic representations $\rho \in R(J)$ is nondegenerate. This was addressed in the proof of [SZ17, Theorem 4.8]:

Lemma 4.8. Let

 $\rho: \pi_1(L \setminus N(J)) \to SU(2)$

be a representation with cyclic image, satisfying $\rho(\mu_J) = i$. If

 $\dim_{\mathbb{R}} H^1(L \setminus N(J); \mathrm{ad}(\rho)) = 1,$

then ρ is nondegenerate as a generator of KHI(L, J).

Proof. The proof of [SZ17, Theorem 4.8] establishes this for knots in S^3 , in the further situation where we have introduced some holonomy perturbation Φ and replaced the condition $\rho(\mu_J) = i$ with a more general condition described by some curve C' in the pillowcase orbifold (i.e., the SU(2) character variety of T^2). The current setting is simpler, though: we take $\Phi = 0$ and the original condition $\rho(\mu_J) = i$, and then the proof works verbatim. \Box

Proposition 4.9. Suppose $H_1(L)$ has odd order and $J \subset L$ is primitive. Let

$$\rho: \pi_1(L \setminus N(J)) \to SU(2)$$

be a representation with cyclic image, satisfying $\rho(\mu_J) = i$. If the $2|H_1(L)|$ -fold cyclic cover of $L \setminus N(J)$ has first Betti number 1, then ρ is nondegenerate as a generator of KHI(L, J).

Proof. Let $C \subset Z$ be as above, so that $[\mu_C]$ generates $H_1(L \setminus N(J)) \cong \mathbb{Z}$ and

$$[\mu_J] = |H_1(L)| \cdot [\mu_C].$$

Since ρ has cyclic image, it follows that $\operatorname{ad} \rho$ has cyclic image as well and therefore factors through $H_1(L \setminus N(J))$. Moreover, $\operatorname{Im}(\operatorname{ad} \rho)$ has order n = 2r for some divisor r of $|H_1(L)|$, because it sends $|H_1(L)| \cdot [\mu_C] = [\mu_J]$ to the order-2 element $\operatorname{ad}(i)$.

For each divisor d of n, let Y_d be the d-fold cyclic cover of $L \setminus N(J)$ corresponding to the index-d subgroup

$$\ker\left(\pi_1(L\setminus N(J))\xrightarrow{\operatorname{ad}\rho}SO(3)\xrightarrow{x\mapsto x^{n/d}}SO(3)\right)$$

Boyer and Nicas [BN90, Theorem 1.1 and Remark 1.2] proved that

$$\dim H^1(L \setminus N(J); \operatorname{ad} \rho) = \dim H^1(L \setminus N(J)) + \frac{2}{\varphi(n)} \sum_{d|n} \mu\left(\frac{n}{d}\right) b_1(Y_d),$$

where μ denotes the Möbius function. If we can show that $b_1(Y_d) = 1$ for all $d \mid n$, then the sum on the right will be

$$\sum_{d|n} \mu\left(\frac{n}{d}\right) = 0$$

since $n \ge 2$, and so we will have dim $H^1(L \setminus N(J); ad \rho) = 1$ as desired.

Note for each $d \mid n$ that Y_n is a finite cover of Y_d , which in turn is a finite cover of $L \setminus N(J)$. It follows by a transfer argument that their first Betti numbers satisfy

$$b_1(Y_n) \ge b_1(Y_d) \ge b_1(L \setminus N(J)) = 1,$$

so if $b_1(Y_n) = 1$ then we will have $b_1(Y_d) = 1$ for all $d \mid n$ after all. Similarly, since n divides $2|H_1(L)|$, we conclude that if the $2|H_1(L)|$ -fold cyclic cover of $L \setminus N(J)$ has $b_1 = 1$ then we also have $b_1(Y_n) = 1$, and hence ρ is nondegenerate.

We now put together the various results of this section to prove Theorem 4.1.

Proof of Theorem 4.1. Proposition 4.7 says that every representation

$$o: \pi_1(L \setminus N(J)) \to SU(2)$$

with $\rho(\mu_J) = i$ has cyclic image. By Proposition 4.5 there are $|H_1(L)|$ such ρ . Thus, if they are all nondegenerate, then we will have

$$\dim_{\mathbb{C}} KHI(L, J) \le |R(J)| = |H_1(L)|,$$

as in (4.1). In the opposite direction, we know that

$$|H_1(L)| \le \dim_{\mathbb{C}} I^{\#}(L) \le \dim_{\mathbb{C}} KHI(L,J)$$

by [LY20, Theorem 1.2], so equality will follow. It remains to establish the nondegeneracy of these cyclic representations; by Proposition 4.9 it will suffice to show that the $2|H_1(L)|$ -fold cyclic cover of $L \setminus N(J)$ has first Betti number 1.

Let Y be the n-fold cyclic cover of $L \setminus N(J) \cong Z \setminus N(C)$, where $n = 2|H_1(L)|$. We can Dehn fill Y along a lift $\tilde{\mu}$ of μ_C^n to get the n-fold branched cyclic cover of C, and this Dehn filling decreases the first Betti number by 1 since $[\tilde{\mu}]$ is a non-torsion element of $H_1(Y)$. (Indeed, we can lift a Seifert surface for C to Y to get a surface whose intersection pairing with $\tilde{\mu}$ is 1.) It thus follows that

$$b_1(Y) = 1$$
 if and only if $b_1(\Sigma_n(Z, C)) = 0$.

But a theorem of Fox [BZ03, Theorem 8.21] says that

$$|H_1(\Sigma_n(Z,C))| = \left|\prod_{k=1}^{n-1} \Delta_C(e^{2\pi i k/n})\right|,$$

where $\Sigma_n(Z, C)$ is the *n*-fold cyclic branched cover of $C \subset Z$. It follows that $b_1(\Sigma_n(Z, C)) = 0$ if and only if no nontrivial *n*th root of unity is a zero of the Alexander polynomial $\Delta_C(t)$.

We now wish to show that the cyclotomic polynomials $\Phi_d(t)$ do not divide $\Delta_C(t)$ for all divisors $d \geq 2$ of $n = 2|H_1(L)|$. If $|H_1(L)|$ is an odd prime power, say p^e , then either

- d = 2, and $\Phi_2(t) = 1 + t$ implies that $\Phi_d(1) = 2$;
- $d = p^k$ with $1 \le k \le e$, and then

$$\Phi_{p^k}(t) = 1 + t^{p^k} + t^{2 \cdot p^k} + \dots + t^{(p-1) \cdot p^k}$$

implies that $\Phi_d(1) = p$; or

• $d = 2 \cdot p^k$ with $1 \le k \le e$, and

$$\Phi_{2 \cdot p^k}(t) = \Phi_{p^k}(-t) = 1 - t^{p^k} + t^{2 \cdot p^k} - \dots + t^{(p-1) \cdot p^k}$$

implies that $\Phi_d(-1) = p$.

Thus, if some $\Phi_d(t)$ divides $\Delta_C(t)$ then at least one of $|\Delta_C(1)|$ and $|\Delta_C(-1)|$ will be strictly greater than 1. But we know that $\Delta_C(1) = 1$ for all knots in homology spheres, and we know from Lemma 4.4 that

$$|\Delta_C(-1)| = |H_1(\Sigma(Z,C))| = 1$$

so we cannot have $\Phi_d(t) \mid \Delta_C(t)$ for any divisor $d \geq 2$ of $2p^e$. We conclude that

$$b_1(\Sigma_n(Z,C)) = 0,$$

proving the required nondegeneracy.

5. FLOER SIMPLE KNOTS AND THEOREM 1.2

Recall that a knot in a lens space L is *simple* if it is isotopic to a union of two arcs, each contained in a meridional disk for one of the two solid tori in a genus-1 Heegaard splitting of L. For instance, the core of each solid torus in such a splitting is a simple knot. Moreover, there is a unique (oriented) simple knot in every homology class in $H_1(L)$. See [Ras07, §2.1] and Section 6 for more on simple knots.

Our main result in this section is Theorem 5.1 below. This theorem is substantially more general than is needed for the proof of Theorem 1.2, but we will use this generality in the next section to extend the latter theorem to additional slopes in the interval [3, 5).

Theorem 5.1. Suppose J is a primitive (oriented) knot in a lens space L such that

$$\dim_{\mathbb{C}} KHI(L,J) = |H_1(L)|.$$

Then J is rationally fibered of the same genus as the simple knot S in its homology class. Moreover, the symmetrized Alexander polynomials of J and S agree,

$$\Delta_J(t) = \Delta_S(t).$$

Finally, if

$$g(J) \le \frac{|H_1(L)| + 1}{4}$$

then J is isotopic to S.

Before proving this theorem, we use it to prove our main result, Theorem 1.2.

Proof of Theorem 1.2. Suppose, for a contradiction, that $S_3^3(K)$ is SU(2)-abelian for some nontrivial knot K. Proposition 3.2 implies that this surgered manifold is homeomorphic to a branched double cover of the lens space

$$L = L(3, 2)$$

along a knot J which is homologous to 5 times a core in a genus-1 Heegaard splitting of L. In particular, the homology class of J (with any orientation) generates

$$H_1(L) \cong \mathbb{Z}/3\mathbb{Z}$$

since gcd(3,5) = 1, meaning that $J \subset L$ is primitive. Its branched double cover is therefore unique, by Lemma 4.2, so we may write

$$S_3^3(K) \cong \Sigma(L, J).$$

Since

$$|H_1(\Sigma(L,J))| = |H_1(L)| = 3,$$

Theorem 4.1 implies that

$$\dim_{\mathbb{C}} KHI(L,J) = |H_1(L)| = 3.$$

Then Theorem 5.1 says that J is rationally fibered and has the same genus as the simple knot in its homology class (for any orientation on J). But each primitive simple knot in L is homologous to, and hence isotopic to, a core of a solid torus in a genus-1 Heegaard splitting of L, and therefore has genus 0. It follows that J has genus 0 as well. The fibered exterior of J is then a solid torus, which implies that its branched double cover $\Sigma(L, J)$ is a lens space. But Proposition 3.2 says that $S_3^3(K)$ is not Seifert fibered, a contradiction. \Box

It remains to prove Theorem 5.1. Before doing so, we review some facts about instanton knot homology and Heegaard knot Floer homology. We will assume for the discussion below that J is a primitive knot in an irreducible rational homology sphere L with $|H_1(L)| = p > 1$. In particular, the complement of J is irreducible.

As in §4, we can identify $L \setminus N(J)$ with $Z \setminus N(C)$ for a knot C in some homology sphere Z. Let

$$(F, \partial F) \subset (L \setminus N(J), \partial N(J))$$

be a genus-minimizing Seifert surface for C, so that F generates $H_2(L \setminus N(J), \partial N(J))$. Then ∂F is a Seifert longitude for C. Since

$$[\mu_J] = |H_1(L)| \cdot [\mu_C] = p[\mu_C] \in H_1(L \setminus N(J))$$

(see §4), ∂F represents a primitive class on $\partial N(J)$, and has algebraic intersection number $\pm p$ with the meridians of J.

Let γ_{μ} be the disjoint union of two oppositely-oriented meridians of J on $\partial N(J)$. Then $(L \setminus N(J), \gamma_{\mu})$ is a balanced sutured manifold, and its sutured instanton homology defines the instanton knot homology of $J \subset L$,

$$KHI(L, J) := SHI(L \setminus N(J), \gamma_{\mu}),$$

as in [KM10b, Definition 7.13]. Let us arrange that ∂F intersects γ_{μ} in 2p points. Then the construction in [LY21b, §2.3] defines an integer-valued Alexander grading on KHI(L, J),

$$KHI(L,J) = \bigoplus_{i \in \mathbb{Z}} KHI(L,J,i).$$

We will review below how this Alexander grading detects genus and fiberedness, and yields a graded Euler characteristic which agrees with that of Heegaard knot Floer homology.

Let

$$n := \frac{1}{2}(p - \chi(F)) = g(F) + \frac{p - 1}{2}.$$

Then [LY21b, Theorem 2.30] says that

$$KHI(L, J, i) = 0 \text{ for } |i| > n,$$

and that if (M_{\pm}, γ_{\pm}) are the sutured manifolds obtained by decomposing $(L \setminus N(J), \gamma_{\mu})$ along $\pm F$, then

$$KHI(L, J, \pm n) \cong SHI(M_{\pm}, \gamma_{\pm}).$$

Since $L \setminus N(J)$ is irreducible with toroidal boundary, and F has minimal genus and minimal intersection with γ_{μ} , the sutured manifolds (M_{\pm}, γ_{\pm}) are both taut. Their sutured instanton homologies are therefore nontrivial [KM10b, Theorem 7.12]; that is,

$$KHI(L, J, \pm n) \neq 0$$

In this way, instanton knot homology detects the genus of J. Note, moreover, that $M_{+} = M_{-}$ and $\gamma_{+} = -\gamma_{-}$. This implies by [LY21b, Theorem 2.30] that the sutured instanton homologies of (M_{\pm}, γ_{\pm}) are isomorphic,

$$KHI(L, J, n) \cong KHI(L, J, -n).$$

The fact that

$$H_2(L \setminus N(J)) \cong H_2(Z \setminus N(C)) = 0$$

implies that $H_2(M_{\pm}) = 0$ as well. We can thus apply [GL19, Theorem 1.2] to conclude that

 $\dim_{\mathbb{C}} KHI(L, J, \pm n) = 1$

iff (M_{\pm}, γ_{\pm}) are product sutured manifolds, or, equivalently, iff J is a rationally fibered knot with fiber surface F. In this way, instanton knot homology detects whether J is rationally fibered. Heegaard knot Floer homology detects genus and fiberedness in the same way.

There is a $\mathbb{Z}/2\mathbb{Z}$ -grading on instanton knot homology which descends to a $\mathbb{Z}/2\mathbb{Z}$ -grading on each Alexander-graded summand KHI(L, J, i), so we can define the graded Euler characteristic of KHI(L, J) by

$$\chi_{\mathrm{gr}}(KHI(L,J)) = \sum_{i \in \mathbb{Z}} \chi(KHI(L,J,i)) \cdot t^i.$$

Recent work of Li and Ye [LY21b, Theorem 1.2] relates this with the graded Euler characteristic of Heegaard knot Floer homology. Namely, they prove that

$$\chi_{\rm gr}(KHI(L,J)) = \pm t^s \cdot \chi_{\rm gr}(\widehat{HFK}(L,J)),$$

for some $s \in \mathbb{Z}$, where the latter is defined with respect to the corresponding Alexander and $\mathbb{Z}/2\mathbb{Z}$ -gradings in the Heegaard Floer setting.

Recall that

(5.1)
$$\chi_{\rm gr}(\widehat{HFK}(L,J))_{t=1} = \chi(\widehat{HF}(L)) = |H_1(L)| = p.$$

Recall further that the Alexander grading on the Heegaard knot Floer homology of J can be viewed as a grading by relative Spin^c structures in $\operatorname{Spin}^c(L, J)$. In the spectral sequence

$$\widehat{HFK}(L,J) \implies \widehat{HF}(L),$$

this grading projects to the Spin^c grading on the latter via the natural map

$$\operatorname{Spin}^{c}(L, J) \to \operatorname{Spin}^{c}(L).$$

These sets of Spin^c structures are affine spaces over the corresponding first homology groups, and the projection map above can be identified with the quotient map

$$H_1(L \setminus N(J)) \cong \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \cong H_1(L)$$

More concretely, there is an identification $\operatorname{Spin}^{c}(L) \cong \mathbb{Z}/p\mathbb{Z}$ such that the spectral sequence above restricts, for each $[i] \in \mathbb{Z}/p\mathbb{Z}$, to a spectral sequence

(5.2)
$$\bigoplus_{m \equiv i \pmod{p}} \widehat{HFK}(L,J,m) \implies \widehat{HF}(L,[i]).$$

In particular, for each $[i] \in \mathbb{Z}/p\mathbb{Z}$, we have

(5.3)
$$\sum_{m \equiv i \pmod{p}} \chi(\widehat{HFK}(L,J,m)) = \chi(\widehat{HF}(L,[i])) = 1$$

The following additional constraints on the graded Euler characteristic of Heegaard knot Floer homology will be important in our proof of Theorem 5.1.

The first lemma below is proved in [Ras07, Corollary 5.3].

Lemma 5.2. Let J_1 and J_2 be two homologous primitive knots in L. Then the difference

$$\chi_{
m gr}(\widehat{H}F\widehat{K}(L,J_1)) - \chi_{
m gr}(\widehat{H}F\widehat{K}(L,J_2))$$

is divisible by $(t^p - 1)^2$.

The next lemma follows immediately from (5.3).

Lemma 5.3. Let J be a primitive knot in L. Then, for each $[i] \in \mathbb{Z}/p\mathbb{Z}$, we have that

$$\chi(\widehat{H}F\widehat{K}(L,J,m)) \neq 0$$

for some integer $m \equiv i \pmod{p}$.

The final lemma below is proved in [Ras07, Proof of Theorem 1]. We only need it for the last claim in Theorem 5.1 in the case where g(J) > 0. We do not need it when g(J) = 0 (in particular, we do not need it for the proof of Theorem 1.2) because we know automatically in this case that J is a core and thus simple.

Lemma 5.4. Let J be a primitive knot in L. Recall that we have assumed that $|H_1(L)| = p$. Suppose in addition that $\dim_{\mathbb{Z}/2\mathbb{Z}} \widehat{HF}(L) = p$ and

$$g(J) < \frac{p+1}{2}.$$

Then $\dim_{\mathbb{Z}/2\mathbb{Z}} \widehat{HFK}(L,J) = p$. That is, J is Heegaard Floer simple.

We may now prove Theorem 5.1.

Proof of Theorem 5.1. Let $J \subset L$ be as in the hypothesis of the theorem, and let S denote the simple knot in the same homology class as J. Simple knots admit Heegaard diagrams for which the knot Floer complex has trivial differential; in particular, they are Heegaard Floer simple.³ Thus,

$$\dim_{\mathbb{Z}/2\mathbb{Z}}\widehat{HFK}(L,S) = p$$

It then follows from Lemma 5.3 and (5.1) that S satisfies the property

(*) for each
$$[i] \in \mathbb{Z}/p\mathbb{Z}$$
, there is a *unique* integer $m \equiv i \pmod{p}$ such that:
 $\chi(\widehat{HFK}(L,S,m)) = 1$, and $\chi(\widehat{HFK}(L,S,k)) = 0$ for all other $k \equiv i \pmod{p}$.

It also follows from Lemma 5.3 that

$$\lim_{\mathbb{Z}/2\mathbb{Z}} \widehat{HFK}(L, S, m) = 0 \text{ or } 1$$

for each $m \in \mathbb{Z}$. Similarly, Lemma 5.3 applied to J, combined with the facts that

$$\dim_{\mathbb{C}} KHI(L,J) = p$$

and that

(5.4)
$$\chi_{\rm gr}(KHI(L,J)) = \pm t^s \cdot \chi_{\rm gr}(HFK(L,J))$$

for some $s \in \mathbb{Z}$, implies that J also satisfies (*), and, additionally, that

$$\dim_{\mathbb{C}} KHI(L, J, m) = 0 \text{ or } 1$$

for each $m \in \mathbb{Z}$. In particular,

$$\dim_{\mathbb{C}} KHI(L, J, \pm n) = 1$$

for

$$n = g(J) + \frac{p-1}{2},$$

which implies that J is rationally fibered. Since

$$KHI(L, J, i) = 0$$

for |i| > n, and likewise for the Heegaard knot Floer homology of J, and since $\chi_{gr}(\widehat{HFK}(L,J))$ is symmetric under the substitution $t \mapsto t^{-1}$, this then implies that s = 0 in (5.4).

We next prove that J and S have the same genus. For this, we claim, as in [Ras07, Proof of Theorem 2], that

(5.5)
$$\overline{\Delta}_J(t) := \chi_{\rm gr}(\widehat{HFK}(L,J)) = \chi_{\rm gr}(\widehat{HFK}(L,S)) =: \overline{\Delta}_S(t).$$

Indeed, the fact that J and S both satisfy property (*) means that we can write

$$\overline{\Delta}_{S}(t) = \sum_{[i] \in \mathbb{Z}/p\mathbb{Z}} t^{n_{[i]}}$$
$$\overline{\Delta}_{J}(t) = \sum_{[i] \in \mathbb{Z}/p\mathbb{Z}} t^{n_{[i]} + p \cdot m_{[i]}}$$

for some integers $n_{[i]} \equiv i \pmod{p}$ and $m_{[i]}$. Since J and S are in the same homology class, Lemma 5.2 says that $(t^p - 1)^2$ divides the difference

$$\overline{\Delta}_J(t) - \overline{\Delta}_S(t) = \sum_{[i] \in \mathbb{Z}/p\mathbb{Z}} t^{n_{[i]}} (t^{p \cdot m_{[i]}} - 1).$$

³Simple knots are also instanton Floer simple, by [LY20]; see also [BLY20].

However, after dividing this difference by $t^p - 1$, we are left with

$$\sum_{\substack{[i]\in\mathbb{Z}/p\mathbb{Z}\\m_{[i]}\neq 0}} t^{n_{[i]}} (1+t^{m_{[i]}}+\cdots+t^{(p-1)\cdot m_{[i]}}).$$

For this sum to be divisible by another factor of $t^p - 1$, it must have 1 as a root. But that is only possible if the sum is trivial—that is, if $m_{[i]} = 0$ for every $[i] \in \mathbb{Z}/p\mathbb{Z}$. This proves that $\overline{\Delta}_J(t) = \overline{\Delta}_S(t)$ as desired.

Combining (5.4) (recall that s = 0) with (5.5), we have therefore shown that

$$\chi_{\rm gr}(KHI(L,J)) = \pm \chi_{\rm gr}(\widehat{H}F\widehat{K}(L,S)).$$

Since the instanton Floer homology of J has dimension at most 1 in each Alexander grading, and likewise for the Heegaard knot Floer homology of S, it follows that for each integer i,

$$KHI(L, J, i) = 0$$
 iff $\widehat{HFK}(L, S, i) = 0.$

This implies that J and S have the same genus, since the Alexander gradings in these two theories detect genus in the same way.

For the claim about Alexander polynomials, Rasmussen proves in [Ras07, §3.7] that, for any primitive knot J in a rational homology sphere L of order p, we have

$$\overline{\Delta}_J(t) = \Delta_J(t) \cdot \frac{t^{p/2} - t^{-p/2}}{t^{1/2} - t^{-1/2}},$$

where $\Delta_J(t)$ is the symmetrized Alexander polynomial of the complement $L \setminus N(J)$. We have shown above that $\overline{\Delta}_J(t) = \overline{\Delta}_S(t)$, from which it follows that $\Delta_J(t) = \Delta_S(t)$ as well.

For the last claim of the theorem, suppose

$$g(J) \le \frac{|H_1(L)| + 1}{4} = \frac{p+1}{4}.$$

Then J is Heegaard Floer simple, by Lemma 5.4. Furthermore, Baker proved in [Bak06, Theorem 1.1] that any knot J satisfying the genus bound above is a 1-bridge knot in L. Finally, Hedden proved in [Hed11, Proposition 3.3] that any 1-bridge knot in a lens space which is also Heegaard Floer simple is simple. Thus, $J \cong S$.

6. Other small surgeries and SU(2)

As mentioned in the introduction, we suspect that the following is true:

Conjecture 6.1. If $K \subset S^3$ is a nontrivial knot, then $S^3_r(K)$ is not SU(2)-abelian for any rational number $r \in [0, 5)$.

Kronheimer–Mrowka proved this conjecture for all $r \in [0, 2]$ in [KM04a]. Baldwin–Sivek proved it in [BS19, Theorem 1.8] for the dense subset of rationals in (2, 3) given by

$$r = p/q \in (2,3),$$

where p, q are coprime and p is a prime power. Indeed, they proved that if $S_r^3(K)$ is SU(2)-abelian in this case, then K has genus 1, and is fibered and strongly quasipositive. The only such knot, the right-handed trefoil, does not admit SU(2)-abelian surgeries in this range.

As alluded to in \$1.2, proving Conjecture 6.1 is much more difficult for the corresponding dense subset of rationals in the interval [3, 5), given by

$$r = p/q \in [3,5),$$

where p, q are coprime and p is a prime power. If $S_r^3(K)$ is SU(2)-abelian in this case, then it is an instanton L-space, by Lemma 3.1, from which it follows that K has genus at most 2, and is fibered and strongly quasipositive [BS19]. One can rule out the trefoil, so K has genus 2, in which case it also has the same Alexander polynomial as $T_{2,5}$, by Lemma 2.2. The difficulty is that there are infinitely many such knots; see, for example, [Mis17].

In this section, we illustrate how our techniques can be used to prove Conjecture 6.1 for infinitely many slopes $r \in [3, 5)$, culminating in the proof of Theorem 1.4.

We start with a comparatively easy case, in which the numerator of r is a power of 2, which is handled with a trick involving a result of Klassen. In this case, we can actually deal with all slopes less than 7.

Theorem 6.2. Let p, q be coprime positive integers with p/q < 7, where p is a power of 2. Then $S^3_{p/q}(K)$ is not SU(2)-abelian for any nontrivial knot $K \subset S^3$.

Proof. Suppose $Y = S_{p/q}^3(K)$ is SU(2)-abelian. Then it is an instanton L-space by Lemma 3.1. It then follows from [BS19, Theorem 1.15] that

$$2g(K) - 1 \le p/q < 7,$$

which implies that $g(K) \leq 3$. If g(K) = 1 then K is the right-handed trefoil, and Y is Seifert fibered with base orbifold $S^2(2,3,\Delta)$ where

$$\Delta = |6q - p|$$

We cannot have $\Delta \leq 1$, because otherwise either $\frac{p}{q} = \frac{6}{1}$ or

$$\frac{p}{q} = 6 \pm \frac{1}{n} = \frac{6n \pm 1}{n}$$

for some integer $n \ge 1$, and in these cases p is not actually a power of 2. But then $\Delta > 1$ and so Y is not SU(2)-abelian [SZ19], a contradiction. Therefore, g(K) is either 2 or 3, which then implies that $p/q \ge 3$. Since p is a power of 2, it must be a multiple of 4.

Theorem 2.1 now tells us that K has Alexander polynomial

$$\Delta_K(t) = t^2 - t + 1 - t^{-1} + t^{-2}$$

if g(K) = 2, whereas if g(K) = 3, then the same argument as in the proof of Lemma 2.2 (appealing to [LY21c, Theorem 1.9] as well as [KM10b, Proposition 7.16] and [BS18a, Theorem 1.7]) says that

$$\Delta_K(t) = t^3 - t^2 + t - 1 + t^{-1} - t^{-2} + t^{-3}$$

or $t^3 - t^2 + 1 - t^{-2} + t^{-3}$.

In each of these cases we have

$$det(K) = |\Delta_K(-1)| = 5 \text{ or } 7 \text{ or } 3,$$

respectively. From here we argue exactly as in [BS19, Lemma 9.5]: Klassen [Kla91, Theorem 10] proved that there are

$$\frac{\det(K) - 1}{2} = 2 \text{ or } 3 \text{ or } 1$$

respectively, conjugacy classes of non-abelian representations

$$\rho: \pi_1(S^3 \setminus K) \to SU(2)$$

with image in the binary dihedral group $D_{\infty} = \{e^{i\theta}\} \cup \{e^{i\theta}j\}$. Let μ be a meridian of K and λ the Seifert longitude. Following the proof of [Kla91, Theorem 10], since ρ has non-abelian image we must have that $\rho(\mu) = j$ up to conjugacy, and hence $\rho(\mu^4) = 1$. Moreover, we know that λ lies in the second commutator subgroup of $\pi_1(S^3 \setminus K)$; thus $\rho(\lambda)$ lies in the second commutator subgroup of $\pi_0(\lambda) = 1$. Since p is a multiple of 4, it follows that $\rho(\mu^p \lambda^q) = 1$, so ρ descends to a representation

$$\pi_1(S^3_{p/q}(K)) \to SU(2)$$

with non-abelian image, but we assumed $Y = S^3_{p/q}(K)$ is SU(2)-abelian, a contradiction. \Box

We next turn to the much more difficult case in which the numerator of r is an odd prime power; this will require the full strength of the techniques in this paper. To start, we review some facts about simple knots in lens spaces, and then study these knots in greater depth.

If $A \subset S^3$ is an unknot, then the lens space L(a, b) is given in our notation by

$$L(a,b) = S^3_{a/b}(A).$$

The standard genus-1 Heegaard splitting of this lens space is given by the union of the solid torus exterior $S^3 \setminus N(A)$ with the surgery solid torus. The core of each solid torus generates the first homology of the lens space. Let S(a, b, k) denote the unique (oriented) simple knot which is homologous to k times the core of the surgery solid torus, for some orientation of this core.⁴ Note that

$$S(a,b,k) \cong S(a,b,k')$$

when $k \equiv k' \pmod{a}$. Moreover, $S(a, a - b, k) \subset -L(a, b)$ is the orientation-reverse of the mirror of S(a, b, k). Finally, $S(a, b, \pm 1)$ is isotopic to a core and therefore has genus 0.

Several of our results thus far, like Proposition 3.2, pertain to a knot $J \subset L(p, 2q)$ which is homologous to 5 times a core of the *other* solid torus $S^3 \setminus N(A)$ in the genus-1 splitting of this lens space. It is easy to check (for some orientation on J) that

$$[J] = [S(p, 2q, 10q)] \in H_1(L(p, 2q)).$$

Note that when p is odd, the knot S(p, 2q, 10q) is primitive if and only if gcd(p, 5) = 1.

The main theorem of this section is as follows:

Theorem 6.3. Let p and q be coprime positive integers with $p/q \in [3,5)$, where p is an odd prime power and gcd(p,5) = 1. If either $p/q \in (4,5)$ or

$$g(S(p, 2q, 10q)) \le \frac{p+1}{4},$$

then $S^3_{p/q}(K)$ is not SU(2)-abelian for any nontrivial knot $K \subset S^3$.

⁴This simple knot is denoted by K(a, b, k) in [Ras07].

Before proving this theorem, we establish the following proposition. In addition to helping us prove Theorem 6.3, this proposition also provides the genera of the relevant simple knots S(p, 2q, 10q), making it easy to apply Theorem 6.3 in practice. Indeed, we will apply this genus formula at the end to prove Theorem 1.4.

Proposition 6.4. Let p and q be coprime positive integers with $p/q \in [3, 6]$, where p is odd and gcd(p, 5) = 1. Then the branched double cover of

$$S = S(p, 2q, 10q) \subset L(p, 2q) = L$$

is Seifert fibered with

$$|H_1(\Sigma(L,S))| = \begin{cases} p, & p/q \in [3,4) \\ 5p, & p/q \in (4,6). \end{cases}$$

Moreover, this primitive simple knot has genus

$$g(S) = ||4p - 10\min\{2q, p - 2q\}| - 2|.$$

Proof. Let a, b be coprime positive integers with $b/a \in [1/3, 2/3]$ and gcd(a, 5) = 1; eventually we will take (a, b) = (p, 2q). We will attempt to understand the branched double cover and genus of the primitive simple knot

$$S(a, b, 5b) \subset L(a, b).$$

Without loss of generality, we can assume that

$$b/a \in [1/3, 1/2]$$

Indeed, if $b/a \in [1/2, 2/3]$ then $(a-b)/a \in [1/3, 1/2]$, and we can just consider S(a, a-b, 5b) instead, as it is the orientation-reverse of the mirror of S(a, b, 5b).

In [Ye20, §8], Ye determines when a simple knot in L(a, b) is isotopic to a torus knot on the Heegaard torus in the genus-1 splitting of the lens space. More precisely, let us think of L(a, b) as obtained via (a/b)-surgery on the unknot $A \subset S^3$, with genus-1 Heegaard splitting given by the union of

$$H_{\alpha} = S^3 \setminus N(A)$$

with the surgery solid torus H_{β} , as above. In particular, the meridian of H_{β} is glued along the curve $a\mu + b\lambda$, with respect to a meridian μ and Seifert longitude λ of A on ∂H_{α} , with $\mu \cdot \lambda = -1$. Let

$$\Sigma = \partial H_{\alpha} = -\partial H_{\beta}$$

be the Heegaard torus in this splitting. Let T be the (5, 2)-torus knot on Σ —that is, the simple closed curve on Σ which is homologous in Σ to $5\mu + 2\lambda$. Since $b/a \in [1/2, 1/3]$, [Ye20, Theorem 8.2] says that

$$S(a, b, 5b) \cong T.$$

Indeed, following the discussion above, it is easy to see that

$$[S(a, b, 5b)] = [T] \in H_1(L(a, b)),$$

so it suffices to show that T is simple. The basic idea behind Ye's result, inspired by [GLV18, §3.1], is that if we fix a Heegaard diagram (Σ, α, β) for the Heegaard splitting

$$L(a,b) = H_{\alpha} \cup_{\Sigma} - H_{\beta}$$

where α is homologous to λ and β is homologous to $a\mu + b\lambda$, then the slope 5/2 is sufficiently close to a/b in this case that we can represent T as the union of an arc in β with an arc in α . Thus, T is simple.

Our goals are then to show that the branched double cover of T is Seifert fibered, compute its genus, and compute the order of its branched double cover.

For the former, note that one can Seifert fiber the solid tori H_{α} and H_{β} so that T is a regular fiber in both. These glue to give a Seifert fibration of L(a, b) with two singular fibers in which T is a regular fiber. The complement of a neighborhood of T is thus Seifert fibered, so the double cover of this complement is as well (each circle fiber lifts to one or two circles). The Seifert fibration on this double cover then extends across the solid torus we glue in to form the branched double cover of T. (This is the same reasoning that shows that torus knots in S^3 have Seifert fibered branched double covers.)

We turn next to the genus calculation. The knot Floer homology of T, and therefore the genus of this knot, is determined by its graded Euler characteristic since simple knots are Floer simple. Let L = L(a, b). Following Rasmussen in [Ras07, §3.7], we have that

$$\chi_{\rm gr}(\widehat{HFK}(L,T)) = \Delta_T(t) \cdot \frac{t^{a/2} - t^{-a/2}}{t^{1/2} - t^{-1/2}},$$

where $\Delta_T(t)$ is the symmetrized Alexander polynomial of T. Letting deg $(\Delta_T(t))$ denote the top degree of this Laurent polynomial, it follows that the top non-zero Alexander grading of $\widehat{HFK}(L,T)$ is given by

$$\deg(\Delta_T(t)) + \frac{a-1}{2}.$$

On the other hand, from the discussion in the previous section, this top Alexander grading is also given by

$$g(T) + \frac{a-1}{2}$$

Therefore,

$$g(T) = \deg(\Delta_T(t)).$$

It thus suffices to compute this Alexander polynomial.

The Alexander polynomial $\Delta_T(t)$ can be computed via Fox calculus, from a presentation of the fundamental group of $L \setminus N(T)$, as described in [Ras07, §3.7]. One finds a 2-generator, 1-relator presentation of this group using Seifert–Van Kampen, exactly as for torus knots in S^3 . Namely, the complement of T is obtained by gluing the solid tori H_{α} and H_{β} together along the annulus on Σ given by

$$C = \Sigma \setminus N(T).$$

Let x and y be the homotopy classes of the cores of H_{α} and H_{β} , respectively. The core of C is homotopic to x^5 in $\pi_1(H_{\alpha})$ and y^d in $\pi_1(H_{\beta})$ (for some orientation of these cores), where

$$d = |\Delta(5/2, a/b)| = |2a - 5b|.$$

Therefore, we have a presentation

$$\pi_1(L \setminus N(T)) = \langle x, y \mid x^5 = y^d \rangle$$

This is the same as the fundamental group of the complement of the torus knot

$$T_{5,d} \subset S^3$$
,

so Fox calculus tells us that $\Delta_T(t)$ agrees with the Alexander polynomial of this torus knot. In particular,

$$g(T) = \deg(\Delta_T(t)) = \deg(\Delta_{T_{5,d}}(t)) = |2(d-1)| = ||4a - 10b| - 2|.$$

Recall that if $b/a \in [1/2, 2/3]$ then we replace b with a - b in this formula. In other words, we have that

$$g(S(a, b, 5b)) = ||4a - 10\min\{b, a - b\}| - 2|.$$

If p, q are as in the hypotheses of this proposition, then letting a = p and b = 2q yields the desired genus formula. (Note that p and 2q are coprime, since p and q are and p is odd.)

Finally, we compute the order of the branched double cover of

$$S = S(p, 2q, 10q) \subset L(p, 2q) = L,$$

where p and q are as in the hypothesis of the proposition. By Lemma 4.4, we have that

(6.1)
$$|H_1(\Sigma(L,S))| = p \cdot |\Delta_S(-1)|.$$

Recall that

$$\Delta_S(t) = \Delta_{T_5 d}(t),$$

for some d, as explained above. If $p/q \in [3,4)$, then $2q/p \in (1/2,2/3]$, and

$$d = |2p - 5(p - 2q)| = |10q - 3p|$$

is odd since p is odd. In this case, an easy calculation shows that $|\Delta_{T_{5,d}}(-1)| = 1$, which implies that $|H_1(\Sigma(L,S))| = p$ by (6.1). If $p/q \in (4,6)$ instead, then $2q/p \in (1/3, 1/2)$, and

$$d = |2p - 10q|$$

is even. In this case, $|\Delta_{T_{5,d}}(-1)| = 5$, and therefore $|H_1(\Sigma(L,S))| = 5p$.

We remark that the formula for g(S(p, 2q, 10q)) from Proposition 6.4 can be simplified for various values of p/q, as follows:

(6.2)
$$g(S(p,2q,10q)) = \begin{cases} 20q - 6p - 2, & 3 \le p/q < 10/3 \\ 6p - 20q - 2, & 10/3 < p/q < 4 \\ 20q - 4p - 2, & 4 < p/q < 5 \\ 4p - 20q - 2, & 5 < p/q \le 6 \end{cases}$$

We have omitted the cases p/q = 10/3, 4, 5 above because of the requirement that p be odd and not a multiple of 5.

Proof of Theorem 6.3. Suppose $Y = S^3_{p/q}(K)$ is SU(2)-abelian for p and q as in the theorem. Proposition 3.2 implies that

$$Y \cong \Sigma(L,J)$$

for some primitive knot J in the lens space L = L(p, 2q), where J (for some orientation) is homologous to the simple knot

$$S = S(p, 2q, 10q),$$

as described above. Proposition 3.2 also implies that Y is not Seifert fibered. From here, Theorem 4.1 tells us that

$$\dim_{\mathbb{C}} KHI(L,J) = |H_1(L)|,$$

and so Theorem 5.1 implies that $\Delta_J(t) = \Delta_S(t)$ and g(J) = g(S).

If $p/q \in (4,5)$, then

$$|H_1(\Sigma(L,J))| = p \cdot |\Delta_J(-1)| = p \cdot |\Delta_S(-1)| = |H_1(\Sigma(L,S))|,$$

by Lemma 4.4. But the latter order is 5p, by Proposition 6.4. This is a contradiction, since $Y = \Sigma(L, J)$ has order p first homology, as (p/q)-surgery on a knot in S^3 .

Finally, if

$$g(S) \le \frac{p+1}{4},$$

then the same is true of the genus of J, in which case J is isotopic to S, by Theorem 5.1. But then $Y \cong \Sigma(L, J) \cong \Sigma(L, S)$ is Seifert fibered, by Proposition 6.4, a contradiction. \Box

Finally, we combine Theorems 6.2 and 6.3 and Proposition 6.4 to prove Theorem 1.4.

Proof of Theorem 1.4. Let p and q be coprime positive integers with $p/q \in [3,7)$, where p is a prime power, as in the hypothesis of the theorem, and let $K \subset S^3$ be a nontrivial knot. We know that

$$Y = S^3_{p/q}(K)$$

is not SU(2)-abelian when p is even, by Theorem 6.2.

Suppose now that p is odd with gcd(p,5) = 1. Let us assume first that $p/q \in [4,5)$. Then Theorem 6.3 says that Y is not SU(2)-abelian. Next, assume that $p/q \in [3,4)$ and

$$23p - 9 \le 80q \le 25p + 9.$$

These inequalities are equivalent to the statement that both 20q - 6p - 2 and 6p - 20q - 2 are at most (p+1)/4. Thus,

$$g(S(p, 2q, 10q)) \le \frac{p+1}{4},$$

by (6.2). Therefore, Theorem 6.3 says that Y is not SU(2)-abelian.

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